Psets 4 - 7 (selected)

3.12 King $\frac{\text{energy dissipated / cycle}}{\text{energy stored}} = \frac{2\pi}{Q}$.

Total $E = Mgh$

cycles $= T / 2\pi\sqrt{\frac{l}{g}} = \frac{T}{2\pi}\sqrt{\frac{g}{l}}$

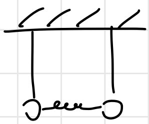
energy stored $= \frac{1}{2} m A^2 \cdot \frac{g}{l}$

$$\therefore Q = \frac{\cancel{2\pi} \cdot \cancel{\tfrac{1}{2}} m A^2 \cancel{g} \cdot T}{l \cdot Mgh \cdot \cancel{2\pi}} \sqrt{\frac{g}{l}}$$

$$= \frac{0.20 \cdot 0.03^2 \cdot 691200 \cdot 9.81^{1/2}}{0.75^{3/2} \; 4.5 \cdot 0.95 \cdot 2} = 70.17 \approx \boxed{70}$$

4.1

a)



from notes: $\omega_1 = \sqrt{\frac{g}{l}}, \quad \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$

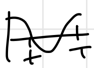
$$\boxed{\omega_1 = 5.72 \text{ rad s}^{-1}, \; \omega_2 = 5.99 \text{ rad s}^{-1}}$$

b) $x_a = \underbrace{A \cos\left(\frac{(\omega_2 - \omega_1)t}{2}\right)}_{\text{modulating}} \cos\left(\frac{(\omega_2 + \omega_1)t}{2}\right)$ from textbook

∴ find first time when this term = 0.

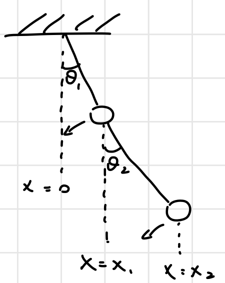
$$T = \frac{2\pi}{\frac{\omega_2 - \omega_1}{2}} = \frac{4\pi}{\omega_2 - \omega_1}$$

$$\therefore t = \frac{1}{4}\frac{4\pi}{\omega_2 - \omega_1} = \frac{\pi}{\omega_2 - \omega_1} = \frac{\pi}{0.17} = \boxed{11.64 \sim 12s}$$



4.5

a)



$T_1 = 2mg$ (heavier)

$T_2 = mg$

$$\therefore m\frac{d^2 x_1}{dt^2} = -T_1 \sin\theta_1 + T_2 \sin\theta_2$$

$$= -2mg\frac{x_1}{l} + mg\frac{(x_2 - x_1)}{l}$$

$$= -\frac{3mgx_1}{l} + \frac{mgx_2}{l}$$

$$\Rightarrow \boxed{\frac{d^2 x_1}{dt^2} + \frac{3gx_1}{l} - \frac{gx_2}{l} = 0}$$

→

$$m\frac{d^2x_2}{dt^2} = -T_2\sin\theta_2$$
$$= -\frac{mg(x_2-x_1)}{l} \Rightarrow \boxed{\frac{d^2x_2}{dt^2} - \frac{g}{l}x_1 + \frac{g}{l}x_2 = 0}$$

b)

$x = C\cos\omega t$ $\qquad x_1: \ -\omega^2 A\cos\omega t = -\frac{3g}{l}A\cos\omega t + \frac{g}{l}B\cos\omega t$

$\dot{x} = -\omega C\sin\omega t$

$\ddot{x} = -\omega^2 C\cos\omega t$ $\qquad x_2: \ -\omega^2 B\cos\omega t = \frac{g}{l}A\cos\omega t - \frac{g}{l}B\cos\omega t$

$$x_1: \ B/A = \frac{3g/l - \omega^2}{g/l}, \quad x^2: \ B/A = \frac{g/l}{3g/l - \omega^2}$$

$AI = \lambda I$:

$$\begin{bmatrix} 3g/l & -g/l \\ -g/l & g/l \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \omega^2\begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{vmatrix} 3g/l - \omega^2 & -g/l \\ -g/l & g/l - \omega^2 \end{vmatrix} = 0$$

$$(3g/l - \omega^2)(g/l - \omega^2) - (g/l)^2 = 0$$
$$\omega^4 - \frac{4g}{l}\omega^2 + \frac{2g^2}{l^2} = 0$$
$$x^2 - 4ax + 2a^2 = 0$$

$$x = \frac{4a \pm \sqrt{16a^2 - 8a^2}}{2} = \frac{4a \pm 2\sqrt{2}a}{2} = (2 \pm \sqrt{2})a$$

$$\Rightarrow \omega^2 = (2 \pm \sqrt{2})a \Rightarrow \boxed{\omega_1 = \sqrt{(2-\sqrt{2})\frac{g}{l}}, \quad \omega_2 = \sqrt{(2+\sqrt{2})\frac{g}{l}}}$$

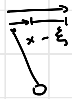
c) $T = 2\pi\sqrt{\frac{l}{g}} \approx 2\,s.$

$T_1 = 2\pi\sqrt{\frac{l}{(2-\sqrt{2})g}} \approx 2.62\,s$

$T_2 = 2\pi\sqrt{\frac{l}{(2+\sqrt{2})g}} \approx 1.08\,s$

French 4.5

$$m\ddot{x} = \underbrace{\frac{mg}{\ell}(x-\xi)}_{F_g\cdot\sin\theta} - \underset{\text{damping}}{\underset{\uparrow}{bv}}$$



$$m\ddot{x} + b\dot{x} + \frac{mg}{\ell}x = \frac{mg}{\ell}\xi$$

$\div$: $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \frac{g}{\ell}x = \frac{g}{\ell}\xi \quad \checkmark$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \omega_0^2\xi_0\cos\omega t.$$

steady state ansatz: $x(t) = \overset{\swarrow A(\omega)}{A}\cos(\omega t - \delta)$

$$x'(t) = -\omega A\sin(\omega t-\delta)$$

$$x''(t) = -\omega^2 A\cos(\omega t-\delta)$$

$\therefore$ $-\omega^2 A\cos(\omega t-\delta) - \gamma\omega A\sin(\omega t-\delta) + \omega_0^2 A\cos(\omega t-\delta) = \omega_0^2\xi_0\cos\omega t$

from notes:

$\Rightarrow$ $\cos\omega t$ terms : $A(\omega)\left[(\omega_0^2-\omega^2)\cos\delta + \gamma\omega\sin\delta\right] = \omega_0^2\xi$

$\sin\omega t$ " : $(\omega_0^2-\omega^2)\sin\delta = \gamma\omega\cos\delta$

$$\Rightarrow \tan\delta = \frac{\gamma\omega}{\omega_0^2-\omega^2} \Rightarrow \delta = \arctan\left(\frac{\gamma\omega}{\omega_0^2-\omega^2}\right)$$

$$A(\omega) = \frac{\omega_0^2\xi}{(\omega_0^2-\omega^2)\cos\delta + \gamma\omega\sin\delta} \Rightarrow \frac{\omega_0^2\xi}{\left[(\omega_0^2-\omega^2) + \omega^2\gamma^2\right]^{1/2}}$$

$$\therefore \boxed{x(t) = \frac{\omega_0^2\xi}{\left[(\omega_0^2-\omega^2)+\omega^2\gamma^2\right]^{1/2}}\cos\left(\omega t - \arctan\left(\frac{\gamma\omega}{\omega_0^2-\omega^2}\right)\right)}$$

b) $\omega = \omega_0$: $Q = \frac{\omega_0}{\gamma}$.

$A(t) = A_0 e^{-\gamma t/2}$. $\quad t = \frac{2}{\gamma}$ in 50 cycles: $\frac{1}{\gamma} = 25\cdot\frac{2\pi}{\omega} = \frac{50\pi}{\omega_0}$

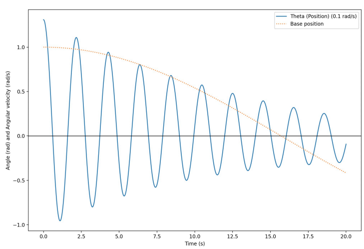
$$A = \frac{\omega_0^2\xi}{\omega\gamma} = Q\xi \Rightarrow A(\omega=\omega_0) = 0.05\pi = \boxed{0.157\text{m}}$$

Numerical:

```python
import math
import numpy as np
import matplotlib.pyplot as plt

# initialize array starting at t=0 w/ 1000 steps, dtype=float
N = 1000
times = np.linspace(0, 20, N + 1)
timestep = 0.02

# assign constants
g = 9.8
l = g/(np.pi)**2
b = 0.15 # damping factor
m = 1 # mass of pendulum
F_0 = 1
gamma = b/m
omega_0 = np.sqrt(g/l)

# initial conditions
theta_0 = 5*np.pi/12    # value from pset (deg converted to radians by *pi/180)
theta_dot_0 = 0    # value from pset
omega = 10   # value from pset

# lists which will be added on to with return values of RK2 function
thetas = [theta_0]
theta_dots = [theta_dot_0]

def RK2(timestep, theta, theta_dot, t):

    # Filling out everything according to formulas
    k1_theta = timestep * theta_dot
    k1_theta_dot = timestep * ((F_0/m * np.cos(omega * t)) - (gamma * theta_dot) - (g/l * np.sin(theta)))

    k2_theta = timestep * (theta_dot + k1_theta_dot/2)
    k2_theta_dot = timestep * ((F_0/m * np.cos(omega * (t + timestep/2))) - (gamma * (theta_dot + k1_theta_dot/2)) - (g/l * np.sin(theta + k1_theta/2)))

    # Update values
    theta_new = theta + k2_theta
    theta_dot_new = theta_dot + k2_theta_dot

    return theta_new, theta_dot_new

theta_current = theta_0
theta_dot_current = theta_dot_0

for i in range(N):
    # assigning return values to current values so the loop can continue
    theta_current, theta_dot_current = RK2(timestep, theta_current, theta_dot_current, times[i])
    # adding values to the list
    thetas.append(theta_current)
    theta_dots.append(theta_dot_current)

# analytical solutions
base = F_0 * np.cos(omega * times)

# table for pset 2a
print("Times, Thetas, Theta_dots:")
for i in range(11):
    print(f"Time: {times[i]}, Theta: {thetas[i]}, Theta_dot: {theta_dots[i]}")

# plotting
plt.figure(figsize=(12, 8))
plt.plot(times, thetas, label=f"Theta (Position) ({omega} rad/s)")
#plt.plot(times, theta_dots, label="Theta_dot (Velocity)", linestyle="-.", color="green")
plt.plot(times, base, label="Base position", linestyle="dotted")
plt.axhline(y=0, color='black', linestyle='-', linewidth=1)
plt.xlabel("Time (s)")
plt.ylabel("Angle (rad) and Angular velocity (rad/s)")
plt.legend()
plt.show()
```

ω:

0.1:




```
Times, Thetas, Theta_dots:
Time: 0.0, Theta: 1.3089969389957472, Theta_dot: 0
Time: 0.02, Theta: 1.3072902778384936, Theta_dot: -0.1704101265517804
Time: 0.04, Theta: 1.3021814007435826, Theta_dot: -0.34013464014618666
Time: 0.06, Theta: 1.2936857754580466, Theta_dot: -0.5089947512290494
Time: 0.08, Theta: 1.2818225309942766, Theta_dot: -0.6767999969005674
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Time: 0.12, Theta: 1.248089896385484, Theta_dot: -1.0083896752451158
Time: 0.14, Theta: 1.2262803119217116, Theta_dot: -1.1716819951779982
Time: 0.16, Theta: 1.2012238715145072, Theta_dot: -1.3329250499425866
Time: 0.18, Theta: 1.1729646872396928, Theta_dot: -1.4917868887759211
Time: 0.2, Theta: 1.1415539066416949, Theta_dot: -1.6478942540266877
```
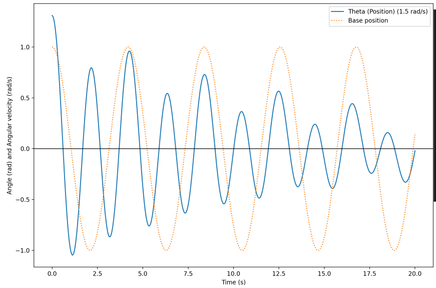

1:




```
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Time: 0.2, Theta: 1.141409920180041, Theta_dot: -1.6508181702047318
```
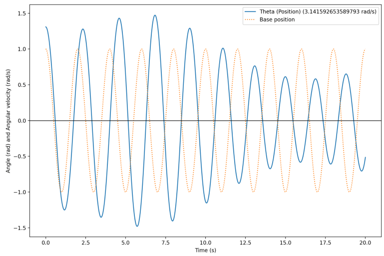

π:




```
Times, Thetas, Theta_dots:
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Time: 0.06, Theta: 1.2936816434120593, Theta_dot: -0.509338555455543
Time: 0.08, Theta: 1.281807996459427, Theta_dot: -0.6776229994009494
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Time: 0.16, Theta: 1.2009681641188628, Theta_dot: -1.339482475898873
Time: 0.18, Theta: 1.1725534973661276, Theta_dot: -1.501079558441667
Time: 0.2, Theta: 1.1409263547450283, Theta_dot: -1.6605693091992144
```
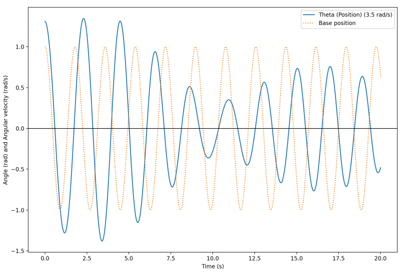

3.5:




```
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Time: 0.04, Theta: 1.302180666935805, Theta_dot: -0.3402568212374589 3
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```
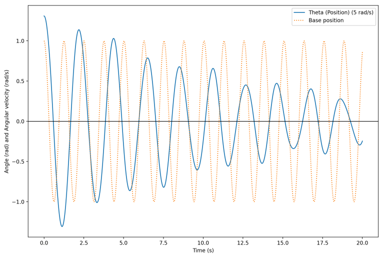

5:




```
Times, Thetas, Theta_dots:
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Time: 0.06, Theta: 1.2936753162090582, Theta_dot: -0.5098639326031559
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```
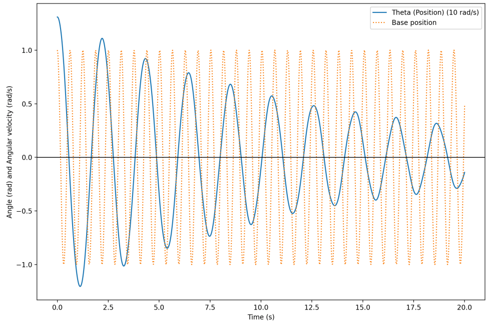

10.




```
Times, Thetas, Theta_dots:
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Time: 0.14, Theta: 1.2248529105555068, Theta_dot: -1.2126370622896656
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Time: 0.18, Theta: 1.1691667898556937, Theta_dot: -1.5731943325177908
Time: 0.2, Theta: 1.1358878120957674, Theta_dot: -1.7551165530285326
```
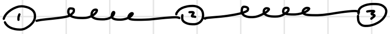

c) resonance frequency $\omega = \omega_0 = \sqrt{\frac{g}{l}} = \sqrt{\frac{g}{\frac{g}{\pi^2}}} = \pi$!

evidently, A is largest. we see A small for $\omega < \pi$, increase to π, then decrease again. This is very pronounced when I extend the time to 100s instead of 20s.
($\gamma^2/2 \ll 1$ so $\omega_{max} \simeq \omega_0$.)

French 5.9 (similar to King 4.6 but I think there's an error here)

a)



$$m_1 \frac{d^2x_1}{dt^2} = -k(x_2 - x_1)$$

$$m_2 \frac{d^2x_2}{dt^2} = -k(x_2 - x_1) + k(x_3 - x_2) = kx_1 - 2kx_2 + kx_3$$

$$m_3 \frac{d^2x_3}{dt^2} = -k(x_3 - x_2)$$

using $x = A\cos\omega t$, $\ddot{x} = -\omega^2 A\cos\omega t$:

m_1: $\omega^2 A\cos\omega t = \frac{k}{m_1}(B\cos\omega t - A\cos\omega t)$

m_2: $\omega^2 B\cos\omega t = k/m_2(-A\cos\omega t + 2B\cos\omega t - C\cos\omega t)$

m_3: $\omega^2 C\cos\omega t = k/m_3(C\cos\omega t - B\cos\omega t)$

$$-\omega^2 \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} k/m & -k/m & 0 \\ -k/M & 2k/M & -k/M \\ 0 & -k/m & k/m \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad (m_1 = m_3 = m,\ m_2 = M)$$

similar to before: solve. (I worked this out elsewhere)

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{k(2m+M)}{mM}}$$

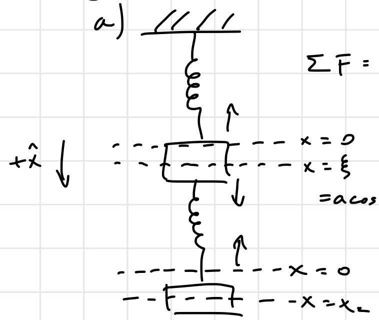
using same method as previous problem:

$$\boxed{\begin{aligned} \omega_1 &\Rightarrow A:B:C = 1:0:-1 \\ \omega_2 &\Rightarrow A:B:C = 1:\frac{-2m_1}{m_2}:1 \end{aligned}}$$

b) $$\frac{\omega_1}{\omega_2} = \frac{\sqrt{\frac{k}{m}}}{\sqrt{\frac{k(2m+M)}{mM}}} = \sqrt{\frac{\cancel{km}M}{\cancel{mk}(2m+M)}} = \sqrt{\frac{M}{2m+M}} = \sqrt{\frac{16}{40}} = \sqrt{\frac{2}{5}} = \boxed{0.63}$$

King 4.9

a)



$$\Sigma F = M\frac{d^2x_1}{dt^2} = F_0\cos\omega t - k_1x_1 + k_2(x_2-x_1)$$

$$\Rightarrow M\frac{d^2x_1}{dt^2} + (k_1+k_2)x_1 - k_2x_2 = F_0\cos\omega t$$

$$m\frac{d^2x_2}{dt^2} = -k_2(x_2-x_1)$$

$$\Rightarrow m\frac{d^2x}{dt^2} + k_2x_2 - k_2x_1 = 0$$

b) $x_1 = A\cos\omega t$, $\ddot{x}_1 = -\omega^2 A\cos\omega t$

$x_2 = B\cos\omega t$, $\ddot{x}_2 = -\omega^2 B\cos\omega t$

x_1: $-M\omega^2 A\cancel{\cos\omega t} + (k_1+k_2)A\cancel{\cos\omega t} - k_2B\cancel{\cos\omega t} = F_0\cancel{\cos\omega t}$

$$(k_1+k_2-M\omega^2)A - k_2B = F_0$$

x_2: $-m\omega^2B\cancel{\cos\omega t} + k_2B\cancel{\cos\omega t} - k_2A\cancel{\cos\omega t} = 0$

$$\Rightarrow A = \left(\frac{k_2-m\omega^2}{k_2}\right)B$$

plug into x_1: $(k_1+k_2-M\omega^2)(k_2-m\omega^2)\frac{1}{k_2}B - k_2B = F_0$

$$\left((k_1+k_2-M\omega^2)(k_2-m\omega^2) - k_2^2\right)B = F_0k_2$$

$$\Rightarrow B = \frac{F_0k_2}{(k_1+k_2-M\omega^2)(k_2-m\omega^2)-k_2^2}$$

$$\text{and } A = B\cdot\left(\frac{k_2-m\omega^2}{k_2}\right) = \frac{F_0(k_2-m\omega^2)}{(k_1+k_2-M\omega^2)(k_2-m\omega^2)-k_2^2}$$

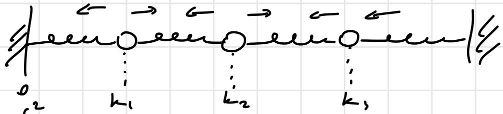
c)

$$A = F_0\left(k_2 - \frac{m}{M}k_1\right) = F_0\left(k_2 - \frac{k_2}{\cancel{k_1}}\cancel{k_1}\right) = 0.$$

numerator $=0$.

4.10

a)



$$m\frac{d^2x_1}{dt^2} = -T_1 + T_2 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$$

$$m\frac{d^2x_2}{dt^2} = -T_2 + T_3 = -k(x_2 - x_1) + k(x_3 - x_2) = kx_1 - 2kx_2 + kx_3$$

$$m\frac{d^2x_3}{dt^2} = -T_3 - \underset{(\text{similar to } T_1)}{T_4} = -k(x_3 - x_2) - k(x_3 - 0) = -2kx_3 + kx_2$$

similar to before: if $x_1 = A\cos\omega t$

$\ddot{x}_1 = -\omega^2 A\cos\omega t$

$$\omega^2\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 2k/m & -k/m & 0 \\ -k/m & 2k/m & -k/m \\ 0 & -k/m & 2k/m \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\begin{vmatrix} 2k/m - \omega^2 & -k/m & 0 \\ -k/m & 2k/m - \omega^2 & -k/m \\ 0 & -k/m & 2k/m - \omega^2 \end{vmatrix} = 0$$

det: $\left((2k/m - \omega^2)^2 - (k/m)^2\right)(2k/m - \omega^2) = -k/m\left(-k/m \cdot (2k/m - \omega^2)\right)$

expand and simplify: $(2k/m - \omega^2)(\omega^4 - 4k/m\,\omega^2 + 2k/m) = 0$

$$\therefore \boxed{\omega^2 = 2k/m \quad \text{or} \quad \omega^2 = (2 \pm \sqrt{2})k/m}$$

b) modes: $A : B : C$

$\omega^2 = 2k/m$: $\cancel{\omega^2 A} = \cancel{\omega^2 A} - \frac{\omega^2}{2}B \Rightarrow A/B = 0$

$\cancel{\omega^2 B} = -\frac{\omega^2}{2}A + \cancel{\omega^2 B} - \frac{\omega^2}{2}C \Rightarrow A = -C$ $\rightarrow$ $\boxed{\begin{matrix} A : B : C \\ = 1 : 0 : -1 \end{matrix}}$

$\omega^2 = (2+\sqrt{2})k/m$: $(2+\sqrt{2})A = 2A - B \rightarrow -\sqrt{2}A = B$

$(2+\sqrt{2})B = -A + 2B - C \Rightarrow \sqrt{2}B = -A - C \Rightarrow$

$(2+\sqrt{2})C = -B + 2C \rightarrow -\sqrt{2}C = B$ $\quad = \boxed{\begin{matrix} A : B : C \\ 1 : -\sqrt{2} : 1 \end{matrix}}$

$\omega^2 = (2-\sqrt{2})k/m$: $(2-\sqrt{2})A = 2A - B \Rightarrow \sqrt{2}A = B$

$(2-\sqrt{2})B = -A + 2B - C \rightarrow \sqrt{2}B = A + C \Rightarrow$

$(2-\sqrt{2})C = -B + 2C \rightarrow \sqrt{2}C = B$ $\quad = \boxed{\begin{matrix} A : B : C \\ 1 : \sqrt{2} : 1 \end{matrix}}$

5.2

$y = A\cos(\omega t - kx)$

$A = 0.15$

$\omega = 2\pi f = 20\pi$

$k = \frac{\omega}{v} = \frac{20\pi}{50} = \frac{2}{5}\pi$

$\therefore y = 0.15\cos(20\pi t - \frac{2}{5}\pi x)\ m$

5.3 a) i) $\frac{\partial y}{\partial t} = A\cos 2\pi\nu(t - x/v)$ $\quad \frac{\partial y}{\partial x} = -\frac{1}{v}A\cos 2\pi\nu(t - x/v)$

$\frac{\partial^2 y}{\partial t^2} = -A\sin 2\pi\nu(t - x/v)$ $\quad \frac{\partial^2 y}{\partial x^2} = -\frac{1}{v^2}A\sin 2\pi\nu(t - x/v)$

$\therefore \frac{\partial^2 y}{\partial x^2}\cdot v^2 = \frac{\partial^2 y}{\partial t^2}$

ii) $\frac{\partial y}{\partial t} = vA\frac{2\pi}{\lambda}\cos(x+vt)$ $\quad \frac{\partial y}{\partial t} = A\frac{2\pi}{\lambda}\cos(x+vt)$

$\frac{\partial^2 y}{\partial t^2} = -v^2 A\frac{2\pi}{\lambda}\sin(x+vt)$ $\quad \frac{\partial^2 y}{\partial t^2} = -A\frac{2\pi}{\lambda}\sin(x+vt)$

$\therefore \frac{\partial^2 y}{\partial t^2}\cdot\frac{1}{v^2} = \frac{\partial^2 y}{\partial x^2}$

iii) $\frac{\partial y}{\partial t} = -\frac{2\pi}{T}A\cos(2\pi(\frac{x}{\lambda} - \frac{t}{T}))$ $\quad \frac{\partial y}{\partial x} = \frac{2\pi}{\lambda}A\cos(2\pi(\frac{x}{\lambda} - \frac{t}{T}))$

$\frac{\partial^2 y}{\partial t^2} = -\underbrace{\left(\frac{2\pi}{T}\right)^2}_{\omega^2}A\sin(2\pi(\frac{x}{\lambda} - \frac{t}{T}))$ $\quad \frac{\partial^2 y}{\partial x^2} = -\underbrace{\left(\frac{2\pi}{\lambda}\right)^2}_{k^2}A\sin(2\pi(\frac{x}{\lambda} - \frac{t}{T}))$

and $\omega^2 = v^2 k^2$ ✓

iv) $\frac{\partial y}{\partial t} = i\omega Ae^{i(\omega t + kx)}$ $\quad \frac{\partial y}{\partial x} = ikAe^{i(\omega t + kx)}$

$\frac{\partial^2 y}{\partial t^2} = -\omega^2 Ae^{i(\omega t + kx)}$ $\quad \frac{\partial^2 y}{\partial x^2} = -k^2 Ae^{i(\omega t + kx)}$

same as above : $\omega^2 = v^2 k^2$

v) $\frac{\partial y}{\partial t} = -\omega_1 A\sin(\omega_1 t - k_1 x) - \omega_2 B\sin(\omega_2 t - k_2 x)$

$\frac{\partial^2 y}{\partial t^2} = -\omega_1^2 A\cos(\omega_1 t - k_1 x) - \omega_2^2 B\cos(\omega_2 t - k_2 x)$

$\frac{\partial y}{\partial x} = k_1 A\sin(\omega_1 t - k_1 x) + k_2 B\cos(\omega_2 t - k_2 x)$

$\frac{\partial^2 y}{\partial x^2} = -k_1^2 A\sin(\omega_1 t - k_1 x) - k_2^2 B\cos(\omega_2 t - k_2 x)$

$\Longrightarrow \omega_1^2 + \omega_2^2 \stackrel{?}{=} v^2(k_1^2 + k_2^2) = \cancel{v^2}\left(\frac{\omega_1^2}{\cancel{v^2}} + \frac{\omega_2^2}{\cancel{v^2}}\right) = \omega_1^2 + \omega_2^2$ ✓

b) $\frac{\partial \psi}{\partial x} = k_1 A\cos(k_1x+k_2y+k_3z-\omega t)$

$\frac{\partial^2 \psi}{\partial x^2} = -k_1^2 A\sin(k_1x+k_2y+k_3z-\omega t)$ similar for y, z.

$\frac{\partial \psi}{\partial t} = -\omega A\cos(k_1x+k_2y+k_3z-\omega t)$

$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 A\sin(k_1x+k_2y+k_3z-\omega t)$

$-(k_1^2+k_2^2+k_3^2) A\sin(k_1x+k_2y+k_3z-\omega t) = -\frac{\omega^2}{v^2} A\sin(\cdots)$

$\therefore k_1^2+k_2^2+k_3^2 = \frac{\omega^2}{v^2}$

$\Rightarrow v = \frac{\omega}{(k_1^2+k_2^2+k_3^2)^{1/2}}$

5.6 a) $f = \frac{c}{\lambda}$

i) $f = \frac{3\times10^8}{1.5\times10^3} = 2\times10^5$ Hz ii) $f = \frac{3\times10^8}{5\times10^{-7}} = 6\times10^{14}$ Hz

iii) $f = \frac{3\times10^8}{10^{-10}} = 3\times10^{18}$ Hz. iv) $f = \frac{3\times10^8\cdot k}{2\pi} = \frac{3\times10^8\cdot 2.1}{2\pi} = \frac{6.3}{6.28}\times10^8$

$\approx 10^8$ Hz

v) $f = \frac{340}{5\times10^{-3}} = 68\times10^3$ Hz ← assume air

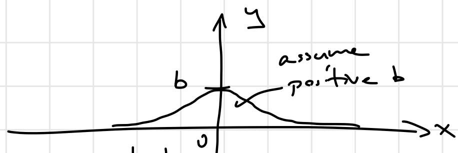
b) $\lambda = \frac{340}{f}$: 20 Hz : $\frac{340}{20} = 17$ m

15 kHz : $\frac{340}{15000} \approx 0.0227$ m.

A = 440 Hz : $\frac{340}{440} \approx 0.773$ m. same order of magnitude as many instruments.

French
7.13

a) $y(x, t=0) = \dfrac{b^3}{b^2 + 4x^2}$



b) as mentioned in my recitation we have something similar to $f(x-vt)$, namely $2x-ut$ in denominator. Let's track points of constant phase, and see what it tells us:

$$2x - ut = \phi = \text{cte.}$$

$\Rightarrow$ $2dx - udt = 0$, or $2dx = udt$

$\Rightarrow$ $\dfrac{dx}{dt} = \dfrac{u}{2}$ [ms^{-1}]. Hence wave travels at $u/2\ ms^{-1}$ in the $+\hat{x}$ direction.

c) $\dfrac{\partial y}{\partial t} = \underset{\text{chain rule}}{\dfrac{\partial}{\partial t}\left(\dfrac{b^3}{g(t)}\right)} = -\dfrac{b^3}{g(t)^2} \cdot \dfrac{dg}{dt} = \dfrac{-b^3}{(b^2+(2x-ut)^2)^2} \cdot \dfrac{\partial}{\partial t}\underbrace{\left(b^2+(2x-ut)^2\right)}_{-2u(2x-ut)}$

$$\therefore \frac{\partial y}{\partial t} = \frac{2b^3u(2x-ut)}{(b^2+(2x-ut)^2)^2}$$

at $t=0$: $\dfrac{4b^3ux}{(b^2+4x^2)^2}$

tells us that motion is symmetric, still for a moment at $x=0$, and the wave is moving particles w/ x coord > 0 up, < 0 back down.

King 6.2

a) $f = 262$, $\mu = 0.04$, $T = 200$

$v = \lambda f$, $v = \sqrt{\frac{T}{\mu}}$ $\Rightarrow$ $\lambda = \sqrt{\frac{T}{\mu}} \cdot \frac{1}{f}$

$= \sqrt{\frac{200}{0.04}} \cdot \frac{1}{262} = \boxed{0.27\,m}$

b) $\lambda = 2L \Rightarrow L = \boxed{0.135\,m}$

c) $v_{wire} = 0.27 \cdot 262 = 70.7\ ms^{-1}$ but $343\,ms^{-1}$ in air.

$v \propto \lambda$ $\therefore$ $\boxed{\lambda \approx 1.32m,\ \nu = 262Hz}$

6.3 a) ratio btw normal modes:

$1, 2 = 2$

$2, 3 = 3/2$

$3, 4 = 4/3$

$4, 5 = 5/4$

Therefore $\boxed{\begin{array}{l} 0.44m \Rightarrow n = 5 \\ 0.55m \Rightarrow n = 4. \end{array}}$

and length is $\boxed{1.1\,m}$ because $n=4$ has $2\lambda = L$.

b) (anti)nodes are 0.5cm apart.

$\therefore \lambda = 1cm$ $\quad \nu = \frac{c}{\lambda} \Rightarrow \frac{3 \times 10^8}{10^{-2}} = \boxed{3 \times 10^{10}\ Hz.}$

6.5 a) $n = 2$: $f_2 = 2f_1 = \boxed{880\ Hz}$

$n = 3$: $f_3 = 3f_1 = \boxed{1320\,Hz}$ wave velocity does NOT change

b) $15000/440 = 34.1 \Rightarrow \boxed{34}$

c) $\lambda \propto f^{-1}$ so $\frac{x}{32} = \frac{440}{523} = 26.9$. so $32 - 26.9 = \boxed{5.1\,cm\ away}$

6.10 a) $n = 1$: $\lambda = 2L = 2m$. $\therefore \nu = \frac{c}{\lambda} = \frac{3 \times 10^8}{2} = 1.5 \times 10^8\ Hz$

$n=2$: 3×10^8, $n=3$: 4.5×10^8

so how many n fit in 4.55×10^9 Hz range?

$\frac{4.55 \times 10^9}{1.5 \times 10^8} \approx \boxed{30}$

b) need base frequency to be 30x as large, so we decrease length.

new $L = 100/30 = \boxed{3.3cm}$

6.15 a) $\lambda_n = \frac{2L}{n} = \frac{h}{p}$

$E = \frac{p^2}{2m} \Rightarrow p = \sqrt{2mE}$

so $\frac{2L}{n} = \frac{h}{\sqrt{2mE}} \Rightarrow \frac{4L^2}{n^2} = \frac{h^2}{2mE}$

$\therefore \boxed{E_n = \frac{n^2h^2}{8mL^2}}$

b) $E_1 = \frac{(6.626\times10^{-34})^2}{8\cdot 9.1\times10^{-31}\cdot(2\times10^{-10})^2}$

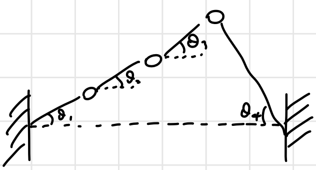
$= \boxed{1.524\times10^{-18}\ J} = 9.5\ eV$

French 6.2

string: $f_n = \frac{v}{\lambda_n}$. $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{TL}{M}}$, $\lambda_n = \frac{2L}{n}$

$\Rightarrow f_n = \sqrt{\frac{TL}{M}}\cdot\frac{n}{2L} = \sqrt{\frac{T}{ML}}\cdot\frac{n}{2}$

normal modes:



$$m_1\frac{d^2y_1}{dt^2} = \frac{-4Ty_1}{L} + \frac{4T(y_2-y_1)}{L} = -\frac{4T}{L}(2y_1 - y_2)$$

$$m_2\frac{d^2y_2}{dt^2} = -\frac{4T(y_2-y_1)}{L} + \frac{4T(y_3-y_2)}{L} = -\frac{4T}{L}(-y_1+2y_2-y_3)$$

$$m_3\frac{d^2y_3}{dt^2} = \frac{-4T(y_3-y_2)}{L} - \frac{4Ty_3}{L} = \frac{-4T}{L}(-y_2+2y_3)$$

assume $y_1 = Ae^{i\omega t}$, $y_2 = B\cdots$ $y_3 = C\cdots$

$\ddot{y} = -\omega^2 Ae^{i\omega t}$

1: $-m_1\omega^2 A\cancel{e^{i\omega t}} = -\frac{4T}{L}\cancel{e^{i\omega t}}(2A-B)$

2: $-m_2\omega^2 B\cancel{e^{i\omega t}} = \frac{-4T}{L}\cancel{e^{i\omega t}}(-A+2B-C)$

3: $-m_3\omega^2 C\cancel{e^{i\omega t}} = \frac{-4T}{L}\cancel{e^{i\omega t}}(-B+2C)$

$-\frac{4T}{L} \div -\frac{M}{3}$

$= \frac{12T}{LM}$

$$\therefore \omega^2\begin{bmatrix}A\\B\\C\end{bmatrix} = \begin{bmatrix}\frac{24T}{LM} & -\frac{12T}{LM} & 0\\ -\frac{12T}{LM} & \frac{24T}{LM} & -\frac{12T}{LM}\\ 0 & -\frac{12T}{LM} & \frac{24T}{LM}\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix}$$

$$\det:\begin{vmatrix}\frac{24T}{LM}-\omega^2 & -\frac{12T}{LM} & 0\\ -\frac{12T}{LM} & \frac{24T}{LM}-\omega^2 & -\frac{12T}{LM}\\ 0 & -\frac{12T}{LM} & \frac{24T}{LM}-\omega^2\end{vmatrix} = 0$$

let $\frac{T}{LM} = x$, $\omega^2 = a$

$$(24x-a)\left[(24x-a)^2-(12x)^2\right] + 12x\left[(-12x)\cdot(24x-a)\right] = 0$$

$$a^3 - 72a^2x + 1440ax^2 - 6912x^3 = 0$$

$$(a-24x)(a^2-48ax+288x^2) = 0$$

1: $a = 24x \Rightarrow \omega^2 = \frac{24T}{LM}$

2: $a = (24+12\sqrt{2})x \Rightarrow \omega^2 = \frac{(24+12\sqrt{2})T}{LM}$

3: $a = (24-12\sqrt{2})x \rightarrow \omega^2 = \frac{(24-12\sqrt{2})T}{LM}$ $\qquad \omega = 2\pi f \Rightarrow f = \frac{\omega}{2\pi}$.

string:

$f_1 = \frac{1}{2}\sqrt{\frac{T}{ML}}$

$f_2 = \sqrt{\frac{T}{ML}}$

$f_3 = \frac{3}{2}\sqrt{\frac{T}{ML}}$

masses:

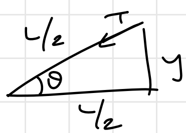
$f_1 = \frac{1}{2\pi}\sqrt{\frac{(24-12\sqrt{2})T}{ML}} \approx 0.42\sqrt{\frac{T}{ML}} \approx 0.84 f_{1,\text{string}}$

$f_2 = \frac{\sqrt{24}}{2\pi}\sqrt{\frac{T}{ML}} = \frac{\sqrt{6}}{\pi}\sqrt{\frac{T}{ML}} \approx 0.78\sqrt{\frac{T}{ML}} \approx 0.78 f_{2,\text{string}}$

$f_3 = \frac{1}{2\pi}\sqrt{\frac{(24+12\sqrt{2} T}{LM}} \approx 1.0\sqrt{\frac{T}{ML}} \approx 0.67 f_{3,\text{string}}$

we see that f_2 string $\sim f_3$ masses The harmonic modes for string increase faster in frequency, and they are integer multiples. They have diff. degrees of freedom, hence, it makes sense that the frequencies don't line up.

6.12 a) consider:



F_y required to hold string is $T\sin\theta \approx T\frac{y}{L/2} = \frac{2Ty}{L}$

$$E = \int_0^h F_y\,dy = \frac{2T}{L}\underbrace{\int_0^h y\,dy}_{\frac{h^2}{2}}$$

$$= \frac{Th^2}{L}$$

same on other side: $\therefore \boxed{E = \frac{2Th^2}{L}}$

b) use fundamental frequency. Because this is slowest.

$v = \lambda f = \frac{\lambda}{t}$

$t = \frac{\lambda}{v} = \frac{2L}{\sqrt{\frac{T}{M}}} = 2L\sqrt{\frac{M}{LT}} = \boxed{2\sqrt{\frac{ML}{T}} \text{ seconds}}$